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Composition of interactions in relativistic quantum theory

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Abstract. A scheme of composition of interactions, which is the generalisation of Sokolov's method of packing operators, is proposed. This scheme can be used in all forms of relativistic quantum dynamics. In contrast to the formulation of Coester and Polyzou, we make no use of the properties of multichannel wave operators, but use only the properties of the Poincaré group unitary representations. A solution to the problem of composition of interactions in the instant form is given, and it is shown that this solution agrees with that obtained by Coester and Polyzou.

1. The statement of the problem

In relativistic quantum theory, a description of all interactions in the considered system is accomplished by assuming some unitary representation of the Poincaré group, the objects comprising the system being in this case either elementary particles or particle fields of a given type.

By definition, a field of particles of a given type is described by a Fock column, the nth component of which is a wavefunction of the state comprising n non-interacting particles of a given type. Accordingly, it is assumed that the wavefunction of a field of particles of a given type is transformed over the Poincaré group representation which is a direct sum of one-particle, two-particle, etc, representations.

Let the considered system comprise N objects $1, 2, \ldots, N$. Hereafter, it is of no importance for us whether these objects are particles or fields of particles of a given type. The only matter of significance is that each object i is described by some unitary representation of the Poincaré group $g \to U_i(g)$ on the Hilbert space $\mathscr{H}_i$.

By definition, the objects $1, 2, \ldots, N$ are not interacting with each other, if the wavefunction of the considered system is transformed over a tensor product of representations $g \to U_i(g)$, $i = 1, 2, \ldots, N$, taking the statistics into account. That is, if there are identical particles among the objects, the system wavefunction should be symmetrised adequately. A representation describing a system of N non-interacting objects will be denoted by $g \to U(g)$. However, if the objects are interacting with each other, then it is assumed that the representation space $\mathscr{H}$ remains the same as for the representation $g \to U(g)$, but operators of the representations $g \to \hat{U}(g)$ differ from those of the representation $g \to U(g)$. Of course, if we consider some subsystem of the considered system (comprising, evidently, not more than $N-1$ objects), then one may speak of the interaction of objects in this subsystem in an analogous way.

It is often convenient to work in physics with representations not of the Poincaré group but of its Lie algebra. In this case it is always assumed that one may construct a representation of the group Lie algebra using the group representation, and *vice* *versa*, the considered representation of the Lie algebra can be uniquely integrated to a global representation of the group. We shall adopt this assumption as well.

As follows from the definition of a tensor product of the group representations, the generators of such a representation are equal to the sum of generators any of which acts non-trivially only over the variables of a single object, acting over the variables of other objects as an identical operator. Thus, a standard technique of introducing the interaction implies that into the expressions for generators of a non-interacting system there is introduced the dependence on the so-called interaction operators which act non-trivially over the variables of at least two objects. The interaction operators are not arbitrary, since the new system of generators as well as the original one should satisfy the commutation relations for representation generators of the Poincaré group Lie algebra. In addition, representation generators should not be dependent on how we enumerate the objects in the considered system.

Let a be some partition of the considered system of N objects into n non-interacting subsystems $\alpha_1, \alpha_2, \ldots, \alpha_n$. By definition, interactions between these subsystems are considered as eliminated if all interaction operators acting non-trivially over the variables of at least two of these subsystems are assumed to be equal to zero. It is evident that in this case the wavefunction of the whole system is transformed over the representation $g \to \hat{U}_{\alpha_1 \ldots \alpha_n}(g) \equiv U(g, a)$ equal to a tensor product (taking the statistics into account) of the unitary representations $g \to \hat{U}_{\alpha_i}(g)$ for the subsystems. Let the representations $g \to \hat{U}_\alpha(g)$ be known for all subsystems α of the considered system and $g \to \hat{U}(g)$ be an arbitrary unitary representation of the Poincaré group for the whole system. If for any a the representation operators $g \to \hat{U}(g)$ go over into $\hat{U}_{\alpha_1 \ldots \alpha_n}(g)$ upon elimination of the interaction between the corresponding subsystems, then it is said that the (algebraic) property of cluster separability is satisfied.

In physical applications the property of cluster separability is a necessary but not sufficient condition to fulfil the macrolocality property (see discussion, for example, in the works by Sokolov (1975) and Coester and Polyzou (1982)). In the present work, we shall limit the discussion only to the above algebraic formulation of the cluster separability property. This property is formulated as the condition C1 in Coester and Polyzou (1982). However, to prove that the stronger properties of separability are fulfilled, one should consider specific models.

The problem of composition of interactions can be formulated now as follows. Let the property of cluster separability be fulfilled for any subsystem of the considered system. In which way can a unitary representation of the Poincaré group for the whole system, which fulfils also the cluster separability property, be constructed? As noted hereinafter, the problem of introduction of the interaction into the system is a particular case of the problem considered.

The problem of composition of interactions was initially dealt with in relativistic quantum mechanics, corresponding to a case when all objects in the considered system are particles. To solve this problem, Sokolov (1977, 1978) proposed a so-called method of packing operators. Proceeding from the concepts of these works and from the multichannel scattering theory, Coester and Polyzou (1982) proposed such an approach to the solution of this problem in which only the properties of the Poincaré group unitary representations and the completeness of multichannel wave operators are used rather than the explicit form of wavefunctions. In such a formulation the problem can be considered both in relativistic quantum mechanics and quantum field theory (either local or non-local). Hereinafter (but independently) we shall propose an approach which is similar in a number of ways to that of Coester and Polyzou. Our approach uses only the properties of the Poincaré group unitary representations. It is shown subsequently how our approach works in three basic forms of dynamics described first by Dirac (1949). We show that in the instant form there is a simple class of solutions without the 'packing' of mass operators. We show also that our solution in the instant form agrees with that obtained by Coester and Polyzou (1982).

2. Different decompositions of space $\mathscr{H}$

In this section, when considering the representation $g \to \hat{U}(g)$ over which the wavefunction of the considered system is transformed, we shall assume that along with the conditions in § 1 there are satisfied some other additional conditions. The latter are believed to be rather natural from the point of view of physical applications; however, a rigorous proof of their satisfiability can be realised in specific models only.

Let us consider first the reduction of the representation $g \to \hat{U}(g)$ on a translation group of the conventional three-dimensional Euclidean space. This reduction defines a spectral measure $\Delta \to \hat{E}(\Delta)$ on the R^3-space of three-dimensional momenta. Suppose that this measure is absolutely continuous relative to the conventional Lebesgue measure $\Delta \to \mu(\Delta)$ on R^3. Then, using the formalism of spectral forms proposed by Kato and Kuroda (1971), one can decompose the representation space $\mathscr{H}$ into the direct integral $\int \oplus \hat{\mathscr{H}}(\boldsymbol{p})\, d^3\boldsymbol{p}$.

Condition 2.1. In the space $\mathscr{H}$ there exists such a dense subspace χ that for any ξ, $\eta \in \chi$ and any point $\boldsymbol{p} \in R^3$ there exists the Radon–Nikodym derivative

$$f(\boldsymbol{p}; \xi, \eta) = \lim_{\Delta \to 0} \frac{(E(\Delta)\xi, \eta)}{\mu(\Delta)} \tag{2.1}$$

where (,) is an inner product on $\mathscr{H}$, and the limit is taken over the sets Δ containing the point $\boldsymbol{p}$ and contracting to this point.

Note that the condition 2.1 is somewhat stronger than those in Kato and Kuroda (1971). However, in applications the conditions stronger than 2.1 are generally satisfied. For instance, in analogy with Kato (1977), where applications to the problem of many bodies are considered, one would suppose also that χ has the topology of its own, and the function $f(\boldsymbol{p}; \xi, \eta)$ is (jointly) continuous in $\boldsymbol{p} \in R^3$; $\xi, \eta \in \chi$.

According to the scheme of Kato and Kuroda (1971), $f(\boldsymbol{p}; \xi, \eta)$ at every $\boldsymbol{p}$ is considered as a semi-inner product on χ and induces naturally an inner product on the quotient space $\chi/\mathcal{N}(\boldsymbol{p})$, where $\mathcal{N}(\boldsymbol{p}) = \{\xi : f(\boldsymbol{p}; \xi, \xi) = 0\}$. $\hat{\mathscr{H}}(\boldsymbol{p})$ is defined then as a completion of $\chi/\mathcal{N}(\boldsymbol{p})$. The map $I(\boldsymbol{p})$: $\chi \to \hat{\mathscr{H}}(\boldsymbol{p})$ is defined as a composite of two canonical homomorphisms $\chi \to \chi/\mathcal{N}(\boldsymbol{p}) \to \hat{\mathscr{H}}(\boldsymbol{p})$. The image of χ at the map $I(\boldsymbol{p})$ is, of course, dense in $\hat{\mathscr{H}}(\boldsymbol{p})$. The sets $\{\xi(\boldsymbol{p})\}$, $\xi(\boldsymbol{p}) \in \hat{\mathscr{H}}(\boldsymbol{p})$ are the elements of the space $\int \oplus \hat{\mathscr{H}}(\boldsymbol{p})\, d^3\boldsymbol{p}$, the former fulfilling the property of f-measurability (see the definition in Kato and Kuroda (1971)) and the property

$$\int_{R^3} \|\xi(\boldsymbol{p})\|^2_{\hat{\mathscr{H}}(\boldsymbol{p})}\, d^3\boldsymbol{p} < \infty \tag{2.2}$$

where $\|\ldots\|_{\hat{\mathscr{H}}(\boldsymbol{p})}$ is the norm in $\hat{\mathscr{H}}(\boldsymbol{p})$. The unitary operator $\hat{\pi}$ from $\mathscr{H}$ to $\int \oplus \hat{\mathscr{H}}(\boldsymbol{p})\, d^3\boldsymbol{p}$ is defined on the elements $\xi \in \chi$ as follows: $\hat{\pi}\xi = \{I(\boldsymbol{p})\xi\}$.

We denote $\mathscr{H}(0) \equiv \mathscr{H}(\boldsymbol{p})$ at $\boldsymbol{p}=0$. Our next task is to construct the unitary operator $\hat{\mathscr{U}}(\boldsymbol{p})$ from $\mathscr{H}(0)$ to $\mathscr{H}(\boldsymbol{p})$ at any $\boldsymbol{p}$. We denote

$$\alpha(\boldsymbol{\lambda}) = (\lambda_0 + 1 + \boldsymbol{\lambda\sigma})/[2(1+\lambda_0)]^{1/2} \tag{2.3}$$

where $\lambda_0 = (1+\boldsymbol{\lambda}^2)^{1/2}$, and $\{\boldsymbol{\sigma}\}$ are Pauli matrices. It is well known that $\alpha(\boldsymbol{\lambda}) \in \mathrm{SL}(2, C)$ and defines the purely Lorentz boosts. Let $\hat{M}$ be a mass operator of the system, and $\hat{e}(m)$ its spectral function. We shall always assume that the operator $\hat{M}$ has a strictly positive lower bound. Since the operator $\hat{U}\{\alpha(\boldsymbol{p}/m)\}$ commutes with $\hat{M}$ at all $\boldsymbol{p}$ and m, we can correctly define the integral

$$\hat{\hat{U}}(\boldsymbol{p}) = \int \hat{U}\{\alpha(\boldsymbol{p}/m)\}\,\mathrm{d}\hat{e}(m) \tag{2.4}$$

as a strong limit of the corresponding Riemannian integral sums. Since $\hat{U}\{\alpha(\boldsymbol{\lambda})\}$ is a unitary operator, one can prove in a standard way that the operator (2.4) is also unitary. Let us introduce the operator $\hat{B}(\boldsymbol{p}) = (1+\boldsymbol{p}^2/\hat{M}^2)^{1/4}$ and define a new operator $\hat{U}(\boldsymbol{p}) = \hat{\hat{U}}(\boldsymbol{p})\hat{B}(\boldsymbol{p})$. To proceed with considerations, we need the following.

Condition 2.2. The subspace χ is invariant under the action of operators $\hat{U}(\boldsymbol{p})$, $\boldsymbol{p} \in R^3$ and $\hat{U}(r)$, $r \in \mathrm{SU}(2)$.

One may expect that this condition would be not too limiting. In applications the role of χ is played usually by a space of functions which are sufficiently smooth and rapidly decreasing at infinity. If interaction operators in the momentum representation are integral operators the kernels of which are sufficiently smooth and rapidly decreasing at infinity, then one should expect condition 2.2 to be surely satisfied. However, the rigorous proof of the above statement is not expected to be a simple one, especially in the case of systems having an infinite number of degrees of freedom.

We can now define the operator $\hat{\mathscr{U}}(\boldsymbol{p})$ by the formula

$$\hat{\mathscr{U}}(\boldsymbol{p})h = I(\boldsymbol{p})\hat{U}(\boldsymbol{p})\xi \tag{2.5}$$

if $h = I(0)\xi$. This operator has an inverse

$$\hat{\mathscr{U}}(\boldsymbol{p})^{-1}h' = I(0)\hat{U}(\boldsymbol{p})^{-1}\eta \tag{2.6}$$

if $h' = I(\boldsymbol{p})\eta$. Proceeding from (2.1), (2.4), from the law of group multiplication in the Poincaré group, and from the formula which places the element of the Lorentz group in correspondence with the element of the group $\mathrm{SL}(2, C)$, one may show through conventional calculations that $\|\hat{\mathscr{U}}(\boldsymbol{p})h\|_{\mathscr{H}(\boldsymbol{p})} = \|h\|_{\mathscr{H}(0)}$. Thus, $\mathscr{U}(\boldsymbol{p})$ is indeed a unitary operator from $\mathscr{H}(0)$ to $\mathscr{H}(\boldsymbol{p})$. The requirement of introducing the operator $\hat{B}(\boldsymbol{p})$ is related to the fact, which is well known in the theory of one-particle representations, that $\mathrm{d}^3\boldsymbol{p}/(m^2+\boldsymbol{p}^2)^{1/2}$, rather than $\mathrm{d}^3\boldsymbol{p}$, is a relativistically invariant measure, where m is the mass of the particle.

Let us consider now the reduction of the representation $g \to \hat{U}(g)$ on the group $\mathrm{SU}(2)$. Proceeding from condition 2.2 and the analogy with the proof of the unitarity of operators $\hat{\mathscr{U}}(\boldsymbol{p})$, we can prove that the operators $\hat{u}(r)$ defined by the formula

$$u(r)h = I(0)\hat{U}(r)\xi, \tag{2.7}$$

where $r \in \mathrm{SU}(2)$, $h = I(0)\xi$, define a unitary representation of the group $\mathrm{SU}(2)$ on $\mathscr{H}(0)$. We denote by $\hat{\boldsymbol{j}}$ the generators of such a representation.

Since the mass operator $\hat{M}$ commutes with operators $\hat{E}(\Delta)$, then, according to the von Neumann theorem, in the representation $\mathscr{H}=\int\oplus\hat{\mathscr{H}}(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}$ it is a decomposable operator $\{\hat{M}(\boldsymbol{p})\}$, where $\hat{M}(\boldsymbol{p})$ is the operator on $\hat{\mathscr{H}}(\boldsymbol{p})$. We denote $\hat{m}=\hat{M}(0)$.

Suppose that the set $\{\hat{\mathscr{U}}(\boldsymbol{p})^{-1}\}$ possesses measurability properties which are sufficient for the operator

$$\hat{\mathscr{U}}^{-1}=\int\oplus\hat{\mathscr{U}}(\boldsymbol{p})^{-1}\,\mathrm{d}^3\boldsymbol{p} \tag{2.8}$$

to be a unitary operator from $\int\oplus\hat{\mathscr{H}}(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}$ to $L_2(\boldsymbol{p})\otimes\hat{\mathscr{H}}(0)$, where $L_2(\boldsymbol{p})$ is a space of complex functions of $\boldsymbol{p}$, the squared modulus of which is integrated in R^3 over the measure μ (a detailed description of the theory of direct operatorial integrals is given in Dixmier (1969)). Then the operator $\hat{\mathscr{U}}^{-1}\hat{\pi}$ is a unitary operator from $\mathscr{H}$ to $L_2(\boldsymbol{p})\otimes\hat{\mathscr{H}}(0)$.

In the space $L_2(\boldsymbol{p})\otimes\hat{\mathscr{H}}(0)$ we can directly calculate the representation generators. Since the calculations are carried out in a complete analogy with calculations of the one-particle representation generators, we produce the result at once:

$$\hat{\boldsymbol{P}}=\boldsymbol{p},\qquad \hat{E}=(\hat{m}^2+\boldsymbol{p}^2)^{1/2},\qquad \hat{\boldsymbol{\mathcal{M}}}=-\mathrm{i}\boldsymbol{p}\times\partial/\partial\boldsymbol{p}+\hat{\boldsymbol{j}},$$

$$\hat{\boldsymbol{\mathcal{N}}}=-\mathrm{i}(\hat{m}^2+\boldsymbol{p}^2)^{1/4}\frac{\partial}{\partial\boldsymbol{p}}(\hat{m}^2+\boldsymbol{p}^2)^{1/4}+\frac{\hat{\boldsymbol{j}}\times\boldsymbol{p}}{\hat{m}+(\hat{m}^2+\boldsymbol{p}^2)^{1/2}}. \tag{2.9}$$

Here $\boldsymbol{p}$ denotes the operator of multiplication by $\boldsymbol{p}$, $\hat{\boldsymbol{P}}$ the three-momentum operator, $\hat{E}$ the energy operator, $\hat{\boldsymbol{\mathcal{M}}}$ the rotation generators, and $\hat{\boldsymbol{\mathcal{N}}}$ the generators of the Lorentz boosts. The expression for the operators $\hat{\boldsymbol{\mathcal{N}}}$ is somewhat different from the well known one in the theory of one-particle representations. This is due to the fact that in the latter the measure $\mathrm{d}^3\boldsymbol{p}/(m^2+\boldsymbol{p}^2)^{1/2}$ is usually chosen rather than d^3p (cf the note on operators $\hat{B}(\boldsymbol{p})$).

We denote by $\Gamma^i(\boldsymbol{p};\hat{m},\hat{\boldsymbol{j}})$, $i=1,2,\ldots,10$, the representation generators defined by formulae (2.9). Then, evidently, the representation generators $g\to\hat{U}(g)$ on the space $\mathscr{H}$ have the form

$$\hat{\Gamma}^i=\hat{\pi}^{-1}\hat{\mathscr{U}}\Gamma^i(\boldsymbol{p};\hat{m},\hat{\boldsymbol{j}})\hat{\mathscr{U}}^{-1}\hat{\pi}. \tag{2.10}$$

A transition to the space $L_2(\boldsymbol{p})\otimes\hat{\mathscr{H}}(0)$ is physically associated with the separation of variables into external and internal ones. The external variable is defined by the three-momentum $\hat{\boldsymbol{P}}$, the internal variables being represented by the variables of the space $\hat{\mathscr{H}}(0)$. However, in relativistic theory one may choose as the operators, which describe the motion of the system as a whole, some other sets of the three commuting self-adjoint operators. Accordingly, the decomposition of the space $\mathscr{H}$ will be of another form. We now describe such a decomposition for two more cases.

In the first case, we take the operator of four-velocity $\hat{G}=\hat{P}/\hat{M}$, where $\hat{P}$ is the four-momentum of the system, as an 'external' variable. Since $\hat{G}^2=1$, it is evident that actually only three operators are independent ones, for example, $\hat{\boldsymbol{G}}=\hat{\boldsymbol{P}}/\hat{M}$. A set of operators $\hat{G}$ defines a spectral measure $\Delta\to\hat{E}(\Delta)$ on the upper flank of the hyperboloid $\lambda^2=1$, where λ is the four-vector $\{\lambda_0,\boldsymbol{\lambda}\}$. It is clear that this spectral measure differs from that considered above. However, here and below we shall use the same notations as above to make the analogy between the considered cases more explicit.

As a measure on the hyperboloid we take $\mathrm{d}\mu(\lambda)=\mathrm{d}^3\boldsymbol{\lambda}/(1+\boldsymbol{\lambda}^2)^{1/2}$. Then, in analogy with the decomposition accomplished above, we can define the unitary operator $\hat{\pi}$

which realises $\mathscr{H}$ in the form $\int\oplus\hat{\mathscr{H}}(\lambda)\,d\mu(\lambda)$. We denote $\hat{\mathscr{H}}(0)\equiv\hat{\mathscr{H}}(\lambda)$ at $\boldsymbol{\lambda}=0$. Then, as above, we can construct the unitary operator $\hat{\mathscr{U}}(\lambda)$ from $\hat{\mathscr{H}}(0)$ to $\hat{\mathscr{H}}(\lambda)$. Since the measure $d\mu(\lambda)$ is relativistically invariant, it is sufficient now for constructing the operators $\hat{\mathscr{U}}(\lambda)$ to use the operators $\hat{\mathscr{U}}\{\alpha(\boldsymbol{\lambda})\}$ only, and one would not need operators of the type $\hat{B}(\boldsymbol{p})$. However, we shall see that in the problem of composition of interactions the decomposition $\mathscr{H}=\int\oplus\hat{\mathscr{H}}(\boldsymbol{p})\,d^3\boldsymbol{p}$ has an advantage (and rather substantial!) of its own.

Reducing the operators $\hat{U}(r)$, $r\in SU(2)$ and $\hat{M}$ on $\hat{\mathscr{H}}(0)$, we can, as well as in the above case, define the operators $\hat{\boldsymbol{j}}$ and $\hat{m}$. The calculation of the representation generators yields now the same expression (2.10) where $\hat{\mathscr{U}}=\int\oplus\hat{\mathscr{U}}(\lambda)\,d\mu(\lambda)$, and the operators $\Gamma^i(\boldsymbol{p};\hat{m},\hat{\boldsymbol{j}})$ are replaced with the operators $\Gamma^i(\boldsymbol{\lambda};\hat{m},\hat{\boldsymbol{j}})$ the explicit form of which is as follows:

$$\hat{\boldsymbol{P}}=\hat{m}\boldsymbol{\lambda},\qquad \hat{\boldsymbol{\mathscr{M}}}=-i\boldsymbol{\lambda}\times\partial/\partial\boldsymbol{\lambda}+\hat{\boldsymbol{j}},$$
$$\hat{\boldsymbol{\mathscr{N}}}=-i\lambda_0\,\partial/\partial\boldsymbol{\lambda}+(\hat{\boldsymbol{j}}\times\boldsymbol{\lambda})/(1+\lambda_0). \tag{2.11}$$

One more decomposition of the space $\mathscr{H}$ can be realised, if one assumes that the 'external' variable is defined by the set $\{\hat{P}^1,\hat{P}^2,\hat{P}^+\}$ where the '+' and '−' components of four-vectors are derived by the formulae

$$p^+=(1/\sqrt{2})(p^0+p^3),\qquad p^-=(1/\sqrt{2})(p^0-p^3). \tag{2.12}$$

As a measure on the momentum space, we choose now $d\mu(\boldsymbol{p})=dp^+\,dp^1\,dp^2/p^+$. This measure is relativistically invariant. We denote $\hat{\mathscr{H}}(0)\equiv\hat{\mathscr{H}}(\boldsymbol{p})$ at $p^1=p^2=0$, $p^+=\infty$. To construct the unitary operators $\hat{\mathscr{U}}(\boldsymbol{p})$ from $\hat{\mathscr{H}}(0)$ to $\hat{\mathscr{H}}(\boldsymbol{p})$, one uses the representation operators induced by the generators $\hat{M}^{+-}$ and $\hat{M}^{+l}$, $l=1,2$, where $\hat{M}^{\mu\nu}$, $\mu,\nu=0,1,\ldots,3$ are the Lorentz group representation generators. As for $\hat{M}^{12}$ and $\hat{M}^{-l}$, $l=1,2$, they are the generators of a group which is isomorphic to E(2). By reducing the representation operators of this group on $\hat{\mathscr{H}}(0)$, one can define the operators $\hat{\boldsymbol{j}}$ which satisfy the commutation relations for the spin operators. However, this procedure is not that simple in this case, for it uses a contraction. Explicit formulae readily follow from the work by Lev (1983), and we shall not dwell upon this problem.

Introducing the operator $\hat{\mathscr{U}}=\int\oplus\hat{\mathscr{U}}(\boldsymbol{p})\,d\mu(\boldsymbol{p})$, we shall come, as above, to the formula (2.10) where the form of operators $\Gamma^i(\boldsymbol{p};\hat{m},\hat{\boldsymbol{j}})$ is as follows (see e.g. Lev 1983):

$$\hat{\boldsymbol{P}}_\perp=\boldsymbol{p}_\perp,\qquad \hat{P}^+=p^+,\qquad \hat{P}^-=(\hat{m}^2+\boldsymbol{p}_\perp^2)/2p^+,$$
$$\hat{M}^{+-}=ip^+\frac{\partial}{\partial p^+},\qquad \hat{M}^{+l}=-ip^+\frac{\partial}{\partial p^l},\qquad \hat{M}^{12}=-i\boldsymbol{p}_\perp\times\frac{\partial}{\partial\boldsymbol{p}_\perp}+\hat{j}^3,$$
$$\hat{M}^{-l}=-i\left(p^l\frac{\partial}{\partial p^+}+\hat{P}^-\frac{\partial}{\partial p^l}\right)-\sum_{s=1}^{2}\frac{\varepsilon_{ls}}{p^+}(\hat{m}\hat{j}^s+p^s\hat{j}^3) \tag{2.13}$$

where the index $\perp$ means that a projection of the vector on the plane $\{1,2\}$ is taken, $l=1,2$, $\varepsilon_{12}=-\varepsilon_{21}=1$, $\varepsilon_{11}=\varepsilon_{22}=0$.

As will be noted below, the three considered decompositions are used in the instant, point, and front forms of relativistic dynamics, respectively. The common feature of these decompositions is that from ten representation generators of the Poincaré group we construct the mass operator $\hat{M}$, some three-vector operator $\hat{\boldsymbol{K}}$ which specifies the decomposition $\mathscr{H}=\int\oplus\hat{\mathscr{H}}(\boldsymbol{k})\,d\mu(\boldsymbol{k})$ over some measure μ, a three-parameter family of operators used for the construction of operators $\hat{\mathscr{U}}(\boldsymbol{k})$, and a three-parameter family of operators used for the construction of operators $\boldsymbol{j}$.

3. On the problem of equivalence of different forms of relativistic dynamics

In the general case the interaction operators can be present in all ten representation generators $g \to \hat{U}(g)$. However, when calculating some specific processes, it is desirable to deal with the cases when the interaction operators are present in the least possible number of generators. From this point of view, three forms of relativistic dynamics are considered as the most appropriate ones: the point, instant and front forms. In the instant form, the interaction operators can be present only in the operators of energy and purely Lorentz boosts, and in the point form only in the four-momentum operators. In the front form with a marked third axis, one proceeds first to the '+' and '−' components in accordance with (2.12). Then the interaction is introduced into the operators P^- and M^{-l}, $l=1,2$, and the other generators remain free of the interaction. Thus, in the instant and point forms the interaction operators are included in four generators, and in the front one in three generators. Note, however, that in the front form the interaction terms are inevitably present in the operators of discrete symmetries.

There arises the following basic question: are different forms physically equivalent, or does some form describe physical phenomena more adequately? This problem was put forth by Dirac (1949). One aspect of this problem is discussed below.

Let us consider an arbitrary representation $g \to \hat{U}(g)$ and assume that it specifies the decomposition $\mathscr{H} = \int \oplus \hat{\mathscr{H}}(\boldsymbol{k})\, \mathrm{d}\mu(\boldsymbol{k})$ as described in § 2. We consider also a representation $g \to U(g)$ which describes the same system but in the case when all interactions are eliminated. Suppose that, proceeding from this representation, the decomposition $\mathscr{H} = \int \oplus \mathscr{H}(\boldsymbol{k})\, \mathrm{d}\mu(\boldsymbol{k})$ over the same measure as well as the corresponding operators $\mathscr{U}(\boldsymbol{k})$ and $\boldsymbol{j}$ can be defined. Suppose that we have found a unitary operator A from $\mathscr{H}(0)$ to $\hat{\mathscr{H}}(0)$ such that $A\boldsymbol{j}A^{-1} = \hat{\boldsymbol{j}}$. Then, as follows from (2.10), the operator

$$\mathscr{A} = \hat{\pi}^{-1}\left(\int \oplus \hat{\mathscr{U}}(\boldsymbol{k}) A \mathscr{U}(\boldsymbol{k})^{-1}\, \mathrm{d}\mu(\boldsymbol{k})\right)\pi \tag{3.1}$$

realises the unitary equivalence between the representation generators $g \to \hat{U}(g)$ and operators

$$\hat{\hat{\Gamma}}^i = \pi^{-1}\mathscr{U}\Gamma^i(\boldsymbol{k}; \hat{\hat{m}}, \boldsymbol{j})\mathscr{U}^{-1}\pi \tag{3.2}$$

where $\hat{\hat{m}} = A^{-1}\hat{m}A$. From this formula and from (2.9), (2.11) and (2.13) it follows that for the case of three successive decompositions considered in § 2 we can in this way define the unitary equivalence of the arbitrary representation generators and generators of the representation which is specified in the instant, point and front forms, respectively.

It is well known that the unitary equivalence of two representations does not yet guarantee their physical equivalence. In Sokolov and Shatny (1978) there is considered some explicit description of a system of three interacting particles in the point form, the unitary operators from the point form to the instant and front ones are constructed, and it is shown that these operators do not change the S-matrix. However, in the general case we are not sure as to which natural operator can be a candidate for the role of A, and the problem thus remains open.

4. A general formulation of Sokolov's method of packing operators

We proceed now to the solution of the problem of composition of interactions described in § 1. We shall produce a formulation of Sokolov's method of packing operators, which is applicable in each form of relativistic dynamics. As will be noted below, the unitary operators used for solving the problem have the same form as the operators (3.1).

Thus, suppose that for any partition a into subsystems $\alpha_1 \ldots \alpha_n$ we know the representations $g \to U(g; a)$. Suppose that through this representation we can construct a decomposition into a direct integral $\int \oplus \mathscr{H}(\boldsymbol{k}; a)\, d\mu(\boldsymbol{k})$ over the same measure μ which is used for constructing the decomposition $\int \oplus \mathscr{H}(\boldsymbol{k})\, d\mu(\boldsymbol{k})$ corresponding to the representation $g \to U(g)$. Suppose also that for both decompositions there are satisfied all conditions described in § 2.

The operators $\pi(a), \mathscr{U}(\boldsymbol{k}; a), \boldsymbol{j}(a)$ and $m(a)$ as well as the spaces $\mathscr{H}(\boldsymbol{k}; a)$ at different partitions a will, generally speaking, differ from each other. Let b be some other partition of the considered system into subsystems $\beta_1, \beta_2, \ldots, \beta_q$. Following Coester and Polyzou (1982), we shall adopt the following notation. If $\mathscr{O}$ denotes some operator, then by $\mathscr{O}_b$ we shall denote an operator derived from $\mathscr{O}$ when all interactions between the subsystems $\beta_1, \beta_2, \ldots, \beta_q$ have been eliminated. Such an operation, evidently, can be used not only for the operators but for the spaces $\mathscr{H}(\boldsymbol{k}; a)$ as well. Then, following the same work, we define the intersection $a \cap b$ as a partition into non-interacting subsystems obtained in the case when all interactions between the subsystems $\alpha_1, \ldots, \alpha_n$ as well as between the subsystems $\beta_1, \ldots, \beta_q$ are eliminated. The generators $\Gamma^i(a)$ for the representation $g \to U(g; a)$ evidently satisfy the condition

$$\Gamma^i(a)_b = \Gamma^i(a \cap b), \qquad i = 1, 2, \ldots, 10. \tag{4.1}$$

Since all operators and spaces specified here are completely defined by the generators $\Gamma^i(a)$ and by some fixed measure μ, then the condition analogous to (4.1) is satisfied for the former.

Suppose that we have found the unitary operators $A(a)$ from $\mathscr{H}(0)$ to $\mathscr{H}(0; a)$ which satisfy the conditions

$$A(a)\boldsymbol{j}A(a)^{-1} = \boldsymbol{j}(a), \qquad A(a)_b = A(a \cap b). \tag{4.2}$$

Then, as follows from (2.10),

$$\Gamma^i(a) = \mathscr{A}(a)\tilde{\Gamma}^i(a)\mathscr{A}(a)^{-1} \tag{4.3}$$

where

$$\mathscr{A}(a) = \pi(a)^{-1}\left(\int \oplus \mathscr{U}(\boldsymbol{k}; a)A(a)\mathscr{U}(\boldsymbol{k})^{-1}\, d\mu(\boldsymbol{k})\right)\pi, \tag{4.4}$$

$$\tilde{\Gamma}^i(a) = \pi^{-1}\mathscr{U}\Gamma^i(\boldsymbol{k}; \tilde{m}(a), \boldsymbol{j})\mathscr{U}^{-1}\pi, \tag{4.5}$$

$$\tilde{m}(a) = A(a)^{-1}m(a)A(a). \tag{4.6}$$

From (4.2) and from the abovementioned, it follows that

$$\mathscr{A}(a)_b = \mathscr{A}(a \cap b), \qquad \tilde{m}(a)_b = \tilde{m}(a \cap b). \tag{4.7}$$

It should be remembered that though we do not make concrete any of the dynamics forms, one must keep in mind that in this section we consider the problem of composition of interactions in some specific form. In any form, there exist the interaction-free generators at those i at which $\Gamma^i(\boldsymbol{k}; \hat{m}, \boldsymbol{j})$ does not depend on $\hat{m}$. Thus, if we apply a decomposition into a direct integral, which corresponds to the given form (see the end of § 2), then, as follows from (4.3) and (4.5), at these i the operators $\mathscr{A}(a)$ and $\tilde{\Gamma}^i(a)$ commute with each other.

Suppose that we have constructed an operator $\tilde{m}$ from the operators $\tilde{m}(a)$ and a unitary operator $\mathscr{A}$ from the operators $\mathscr{A}(a)$ such that the following conditions are satisfied:

$$\mathscr{A}_a = \mathscr{A}(a), \qquad \tilde{m}_a = \tilde{m}(a). \tag{4.8}$$

Then the formula

$$\tilde{\Gamma}^i = \mathscr{A}\tilde{\Gamma}^i\mathscr{A}^{-1} \tag{4.9}$$

where

$$\tilde{\Gamma}^i = \pi^{-1}\mathscr{U}\Gamma^i(\boldsymbol{k}; \tilde{m}, \boldsymbol{j})\mathscr{U}^{-1}\pi \tag{4.10}$$

gives a solution to the problem, since the conditions of cluster separability (see § 1) are satisfied due to (4.3). From the above it follows also that the operator $\mathscr{A}$ will commute with $\tilde{\Gamma}^i$ at such i at which the representation generators in a given form are free of the interaction. Therefore, the problem of composition of interactions is solved thus without leaving the frame of the given form.

A solution of the mentioned combinatorial problem of construction of the operators $\tilde{m}$ and $\mathscr{A}$ was given by Sokolov (1977) (see also Coester and Polyzou 1982). As the operator $\tilde{m}$, one can take, for example, the operator

$$\tilde{m} = \sum_{k=2}^{N} (-1)^k(k-1)!\,\tilde{m}_{(k)} + v_N \tag{4.11}$$

where v_N is a fully linked part of the operator $\tilde{m}$, and

$$\tilde{m}_{(k)} = \sum_{\dim a = k} \tilde{m}(a) \tag{4.12}$$

where the summing-up is accomplished over all partitions of the system into k subsystems. The operator v_N acts on $\mathscr{H}(0)$. It should commute with $\boldsymbol{j}$ and not depend on the way the objects $1, 2, \ldots, N$ are enumerated.

One may accordingly construct the operator $\mathscr{A}$; however, in this case as well as in the case of the operator $\tilde{m}$ there is a considerable arbitrariness in the solution of the problem. The most substantial element in the solution of the problem is the finding of operators $A(a)$ which satisfy the conditions (4.2). As seen from formula (4.6), the role of these operators is as follows: they 'pack' the operators $m(a)$ to the operators $\tilde{m}(a)$, and the latter at all a act on the same space $\mathscr{H}(0)$ and commute with the same operator $\boldsymbol{j}$. That is why Sokolov referred to his method as that of packing operators.

5. Composition of interactions in three basic forms of dynamics

In the point form all spaces $\mathscr{H}(0; a)$ are, generally speaking, different. Therefore, at a given λ there will be different operators $\mathscr{U}(\lambda; a)$ as well, though they are all induced by the same operator $U\{\alpha(\lambda)\}$ which does not depend on the interaction. Accordingly, all operators $\boldsymbol{j}(a)$ differ, generally speaking, from $\boldsymbol{j}$. To solve the problem of composition of interactions, one has to find the unitary operators $A(a)$ from $\mathscr{H}(0)$ to $\mathscr{H}(0; a)$ which satisfy the conditions (4.2).

In the front form all spaces $\mathscr{H}(\boldsymbol{p}; a)$ at a given $\boldsymbol{p}$ are identical and coincide with $\mathscr{H}(\boldsymbol{p})$. All operators $\mathscr{U}(\boldsymbol{p}; a)$ are also not dependent on a and coincide with $\mathscr{U}(\boldsymbol{p})$. However, the operators $\boldsymbol{j}(a)$, generally speaking, differ from each other, though they act in the same space $\mathscr{H}(0)$. To solve the problem, one has to find the unitary operators $A(a)$ in $\mathscr{H}(0)$ which satisfy the conditions (4.2).

In the instant form all spaces $\mathscr{H}(\boldsymbol{p}; a)$ are identical at a given $\boldsymbol{p}$ and coincide with $\mathscr{H}(\boldsymbol{p})$; however, the operators $\mathscr{U}(\boldsymbol{p}; a)$ depend on the interaction. Due to the fact that the measure μ in this case is not relativistically invariant, for construction of such operators it is not sufficient to deal only with operators of the purely Lorentz boosts, but one has to introduce the operators $B(\boldsymbol{p}; a) = [1 + \boldsymbol{p}^2/M(a)^2]^{1/4}$ as well. Since $B(\boldsymbol{p}; a)_b = B(\boldsymbol{p}; a \cap b)$, the operators $\mathscr{U}(\boldsymbol{p}; a)$ possess the same property. Since the representation operators of the group SU(2) do not depend on the interaction and all spaces $\mathscr{H}(0; a)$ coincide with $\mathscr{H}(0)$, all operators $\boldsymbol{j}(a)$ are identical and coincide with $\boldsymbol{j}$. Therefore, the problem of composition of interactions in this case can be solved completely, if all operators $A(a)$ are assumed to be equal to unity. Note also that in the front and instant forms, evidently, the relations $\pi(a) = \pi$ are satisfied.

6. Discussion

The variant $A(a) = 1$ in the instant form corresponds to the following procedure. We take the mass operators $M(a)$, reduce them on the space $\mathscr{H}(0)$ and obtain the operators $m(a)$. The latter are introduced into formulae of the type (4.11) and (4.12) without any 'packing'. In the point and front forms, we cannot assume $A(a) = 1$. It does not imply, of course, that these forms are 'inferior' compared with the instant one. However, the solution to the problem of composition of interactions in the point and front forms does not follow from the formulation proper of this problem, since some additional assumptions are needed in this case to find the operators which satisfy the conditions (4.2).

The scheme considered is the most general one also for introduction of the interaction into a system (for example, the term v_N in the formula (4.11)). In particular, this scheme can be used in local theories as well. It is well known that if the Lagrangians of local interactions are not dependent on the derivatives of field functions, and if the energy–momentum tensor is integrated over the hypersurfaces $t = \text{constant}$, we automatically arrive at the instant form. In this case the energy operator is given by the formula (see e.g. Bogoliubov and Shirkov 1976)

$$\hat{E} = E - \sum_n \int \mathscr{L}_{\text{int}}^{(n)}(x)\, \mathrm{d}^3\boldsymbol{x} \tag{6.1}$$

where $\mathscr{L}_{\text{int}}^{(n)}(x)$ are the Lagrangians of different interactions. Reducing this relation on the space $\mathscr{H}(0)$, we note that in the local theory the mass operators are introduced into the law of composition of interactions also without 'packing', their composition into a complete mass operator being accomplished linearly by the formulae (4.11) and (4.12). Therefore, the solution of the problem in this case also corresponds to the choice of $A(a) = 1$. However, the calculations of an explicit form of the 'symmetrised product', which defines the operator $\mathscr{A}$, are not, probably, easy from the technical point of view.

Thus, we may conclude that in the instant form in the case of both local and non-local interactions the problem of composition of interactions is solved without 'packing' of mass operators. As for the point and front forms, 'packing' should be necessarily non-trivial.

One may assume that the results obtained have various applications, in particular in the field of the physics of intermediate energies and quark models. We shall not here discuss this problem in more detail, for it would require substantial efforts and space. Some results concerning the applications will be published by the author elsewhere.

In conclusion, we shall show that the solution of the problem so obtained agrees in the instant form with the solution obtained by Coester and Polyzou (1982). Since in the instant form $\pi(a)=\pi$, one may omit this operator assuming that $\mathscr{H}$ is already realised in the form $\int\oplus\mathscr{H}(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}$. Coester and Polyzou (1982) work in the formalism of the multichannel scattering theory, so they introduce an additional space $\mathscr{H}_f=\int\oplus\mathscr{H}_f(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}$ which is a direct sum of the channel spaces. Coester and Polyzou (1982) interconnect the spaces $\mathscr{H}(\boldsymbol{p})$ and $\mathscr{H}_f(\boldsymbol{p})$ as follows: $\mathscr{H}(\boldsymbol{p})=\exp(-\mathrm{i}\boldsymbol{p}\hat{\boldsymbol{X}})\mathscr{H}(0)$, $\mathscr{H}_f(\boldsymbol{p})=\exp(-\mathrm{i}\boldsymbol{p}\boldsymbol{X}_f)\mathscr{H}_f(0)$, where $\hat{\boldsymbol{X}}$ and $\boldsymbol{X}_f$ are the Newton–Wigner position operators on $\mathscr{H}$ and $\mathscr{H}_f$, respectively. We shall consider these relations in the sense of the formula (2.5):

$$\hat{\mathscr{U}}'(\boldsymbol{p})h=I(\boldsymbol{p})\exp(-\mathrm{i}\boldsymbol{p}\hat{\boldsymbol{X}})\xi, \qquad \mathscr{U}'_f(\boldsymbol{p})h_f=I_f(\boldsymbol{p})\exp(-\mathrm{i}\boldsymbol{p}\boldsymbol{X}_f)\xi_f, \tag{6.2}$$

if $h=I(0)\xi$, $h_f=I_f(0)\xi_f$. Furthermore, Coester and Polyzou (1982) consider two decomposable wave operators from $\mathscr{H}_f$ to $\mathscr{H}$:

$$\Omega=\int\oplus\hat{\Omega}(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}, \qquad \tilde{\Omega}=\int\oplus\tilde{\Omega}(\boldsymbol{p})\,\mathrm{d}^3\boldsymbol{p}, \tag{6.3}$$

where

$$\hat{\Omega}(\boldsymbol{p})=\hat{\mathscr{U}}'(\boldsymbol{p})\hat{\Omega}(0)\mathscr{U}'_f(\boldsymbol{p})^{-1}, \qquad \tilde{\Omega}(\boldsymbol{p})=\mathscr{U}'(\boldsymbol{p})\hat{\Omega}(0)\hat{\mathscr{U}}'_f(\boldsymbol{p})^{-1}, \tag{6.4}$$

the indices $\pm$ of these operators being omitted. Instead of the operator $\mathscr{A}^{-1}$ from (4.4), Coester and Polyzou (1982) solve the problem by means of the operator $B=\tilde{\Omega}\Omega^{\dagger}$ (see formula (3.67) in their work). If one assumes the completeness of wave operators, it follows from (6.3) and (6.4) that

$$B(a)=\int\oplus\mathscr{U}'(\boldsymbol{p})\mathscr{U}'(\boldsymbol{p};a)^{-1}\,\mathrm{d}^3\boldsymbol{p}. \tag{6.5}$$

Thus, we have shown explicitly that the auxiliary space $\mathscr{H}_f$ and wave operators can be eliminated from the answer—the result noted by Coester and Polyzou (1982) in the introduction to their work. Since $\boldsymbol{X}(a)_b=\boldsymbol{X}(a\cap b)$, then from (6.5) it follows that the operators $B(a)$ satisfy the same condition not only in the case of three bodies (as noted by Coester and Polyzou (1982)) but in the general case as well.

Let us demonstrate now that $\mathscr{U}'(\boldsymbol{p};a)=\mathscr{U}(\boldsymbol{p};a)$. Since the Newton–Wigner position operator constucted over the generators (2.9) has a conventional form $\mathrm{i}\,\partial/\partial\boldsymbol{p}$, then it follows from (2.10) that

$$\boldsymbol{X}(a)=\mathscr{U}(a)(\mathrm{i}\,\partial/\partial\boldsymbol{p})\mathscr{U}(a)^{-1}. \tag{6.6}$$

Since $\mathscr{U}(a)^{-1}\{\xi(\boldsymbol{p})\}=\{\eta(\boldsymbol{p})\}$, where $\eta(\boldsymbol{p})=\mathscr{U}(\boldsymbol{p};a)^{-1}\xi(\boldsymbol{p})$ and $I(\boldsymbol{p}')\{\xi(\boldsymbol{p})\}=\xi(\boldsymbol{p}')$, one can easily verify, by substituting (6.6) into (6.2), that the condition $\mathscr{U}'(\boldsymbol{p};a)=$ - $\mathscr{U}(\boldsymbol{p};a)$ is indeed satisfied. Thus, the operator (6.5) does coincide with the operators $\mathscr{A}(a)^{-1}$ (see formula (4.4)) at $A(a)=1$.

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Note added in proof. Recently the author learnt that the solution of the considered problem in the instant form of relativistic quantum mechanics had been obtained also by U Mutze (an article on the work will be published in *Phys. Rev.* D). This solution differs from that obtained by Coester and Polyzou (1982).

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